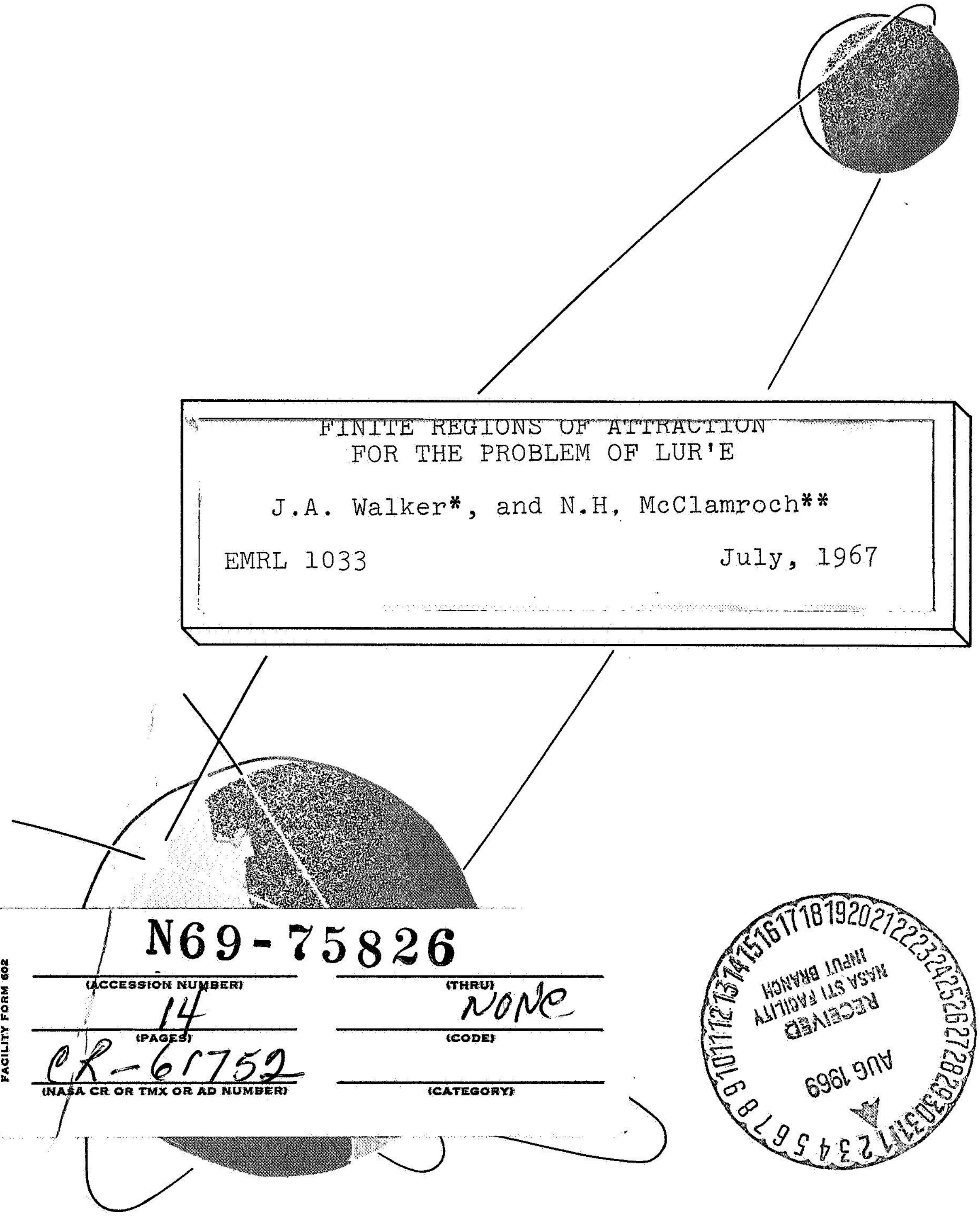

FINITE REGIONS OF ATTRACTION FOR THE PROBLEM OF LUR'E

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Abstract

Finite regions of attraction of the equilibrium are obtained for the problem of Lur'e when the Popov sector condition is satisfied only over a finite or semi-infinite interval. Given this interval, regions of attraction may be specified with or without other knowledge of the nature of the nonlinearity. This specification is performed using Liapunov functions constructed by the Kalman procedure and therefore is completely automatic.

Introduction

The work of Popov (Aizerman and Gantmacher, 1964), concerning control systems involving a single nonlinearity, provides sufficient conditions for global asymptotic stability of the equilibrium when the nonlinearity $\phi(\sigma)$ satisfies a sector condition of the form $0 < \phi(\sigma)/\sigma < K$ for all $\sigma \neq 0$. In many real problems, however, $\phi(\sigma)$ is known to violate this condition for $|\sigma|$ sufficiently large, and in most real problems $\phi(\sigma)$ is not accurately known for large values of $|\sigma|$.

The present work is concerned with the problem of determining a finite region of attraction of the equilibrium of the system

$$\begin{aligned} \dot{\underline{x}} &= A\underline{x} + \underline{b}\phi(\sigma) \\ \sigma &= \underline{c}'\underline{x} \end{aligned} \tag{1}$$

when $\phi(\sigma)$ is a continuous nonlinear function which may leave the sector

$$\begin{aligned} 0 < \phi(\sigma)/\sigma < K \quad , \quad \sigma \neq 0 \\ \phi(0) = 0 \end{aligned} \tag{2}$$

at some $\sigma \leq \ell_1 < 0$ and/or some $\sigma \geq \ell_2 > 0$ but is known to be contained in this sector for all $\sigma \varepsilon (\ell_1, \ell_2)$. Clearly such a region[1] exists if the Popov frequency condition

$$\mathrm{Re}(1 + i\omega q)G(i\omega) + \frac{1}{K} > 0 \qquad \omega \geq 0 \tag{3}$$

is satisfied for some real number q , where

$$G(i\omega) = \underline{c}'[A - i\omega I]^{-1}\underline{b} \tag{4}$$

and A is strictly Hurwitz.

Two types of regions of attraction may be envisaged: a) a region specifically tailored to the particular $\phi(\sigma)$ under consideration, and b) a region which is valid for <u>all</u> nonlinearities satisfying the sector condition for $\sigma \varepsilon (\ell_1, \ell_2)$. Both types of regions will be determined in the following.

Formulation of the Problem

It will be assumed that i) all roots of $|A-zI| = 0$ have negative real parts, ii) the pair $(A,\underline{b})$ is completely controllable, iii) the pair $(\underline{c}',A)$ is completely observable, iv) the Popov condition (3) is satisfied for some q such that $|qA + I| \neq 0$. As stated or implied by several sources (Aizerman and Gantmacher, 1964; Kalman, 1963), these conditions imply the existence of a function $V(\underline{x})$ of the form

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[1]The numbers ℓ_2 and $-\ell_1$ are not both infinite.

$$V(\underline{x}) = \underline{x}'B\underline{x} + q\int_0^{\sigma} \phi(\xi)d\xi \tag{5}$$

having the derivative according to (1)

$$-\dot{V} = [\underline{u}'x-\sqrt{\gamma}\,\phi(\sigma)]^2 + \phi(\sigma)[\sigma-\phi(\sigma)/K], \quad (\gamma \triangleq 1/K-q\underline{c}'\underline{b}) \tag{6}$$

where B is a positive definite[1] real symmetric matrix satisfying

$$A'B + BA = -\underline{uu}' \tag{7}$$

and $\underline{u}$ is a real vector which may be specified (Kalman, 1963) by writing the left side of (3) in the form

$$\text{Re}(1+i\omega q)G(i\omega) + \frac{1}{K} = \frac{\theta(i\omega)\theta(-i\omega)}{|i\omega I-A|\;|-i\omega I-A|} \tag{8}$$

and setting

$$\underline{u}'[zI - A]^{-1}\underline{b} \triangleq \sqrt{\gamma} - \frac{\theta(z)}{|zI-A|} \tag{9}$$

where θ is a real polynomial of degree n with leading coefficient $\sqrt{\gamma}$. Further, the complete observability of $(\underline{c}',A)$ implies (Kalman, 1963) that $\dot{V}$ is not identically zero

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[1] See Appendix A

on $[t_1, t_2] (t_2 > t_1)$ for any nontrivial solution of (1) provided $\sigma \varepsilon (\ell_1, \ell_2)$. In addition[2]

$$V > 0 \qquad \forall \sigma \varepsilon (\ell_1, \ell_2) \; , \; |\underline{x}| \neq 0 \tag{10}$$

and

$$-\dot{V} \geq 0 \qquad \forall \sigma \varepsilon (\ell_1, \ell_2) \tag{11}$$

$$\nabla_{\underline{x}} V \neq 0 \qquad \forall \sigma \varepsilon (\ell_1, \ell_2), |\underline{x}| \neq 0 \tag{12}$$

Regions of Attraction

Determination of a region of attraction of the origin proceeds as follows: Define

$$M_i = \min_{\underline{c}'\underline{x}=\ell_i} V(x) \qquad (i=1,2) \tag{13}$$

and note that this minimum takes place at a point $\underline{x}_o$ at which $c'\underline{x}_o = \ell_i$ and $\nabla_{\underline{x}} V(\underline{x}_o)$ is orthogonal to the hyperplane $\underline{c}'\underline{x} = \ell_i$; i.e., $\underline{x}_o = \ell_i B^{-1}\underline{c}/\underline{c}' B^{-1}\underline{c}$. Therefore

$$M_i = \ell_i^2 / \underline{c}' B^{-1} \underline{c} + q \int_0^{\ell_i} \phi(\xi) d\xi \tag{14}$$

and a connected open region $D (0 \varepsilon D)$ is contained in the region of

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[2]See Appendix B

attraction of the origin if $x \varepsilon D$ implies

$$\underline{x}'B\underline{x} + q \int_0^{\underline{c}'\underline{x}} \phi(\xi)d\xi < \min_{i=1,2} M_i \tag{15}$$

where the given $\phi(\sigma)$ satisfies (2) for all $\sigma\varepsilon(\ell_1,\ell_2)$.

Alternatively a connected open region $\widetilde{D}(0\varepsilon D)$ in the region of attraction of the origin, which is valid for <u>all</u> $\phi(\sigma)$ satisfying (2) for $\sigma\varepsilon(\ell_1,\ell_2)$, may be specified by defining

$$\widetilde{M}_1 = \min_{i=1,2} \ell_i^2/\underline{c}'B^{-1}\underline{c} \tag{16}$$

$$\widetilde{M}_2 = \min_{i=1,2} \ell_i^2[qK/2 + 1/\underline{c}'B^{-1}\underline{c}] \tag{17}$$

The region $\widetilde{D}(\widetilde{D} \subsetneq D)$ is then determined by

$$\underline{x}'Bx + q \frac{K}{2}(\underline{c}'\underline{x})^2 < \widetilde{M}_1 \qquad \text{for } q \geq 0 , \tag{18}$$

and by

$$\underline{x}'Bx < \widetilde{M}_2 \qquad \text{for } q \leq 0 . \tag{19}$$

If it is known that $\phi(\sigma)$ is an odd funciton, then a larger connected open region $\tilde{\tilde{D}}(\widetilde{D} \subsetneq \tilde{\tilde{D}} \subsetneq D)$ may be found as

$$\underline{x}'Bx < \widetilde{M}_1 \qquad \text{for } q \geq 0 , \tag{20}$$

and

$$\underline{x}'B\underline{x} + q\,\frac{K}{2}(\underline{c}'\underline{x})^2 < \tilde{M}_2 \qquad \text{for } q \leq 0\,. \tag{21}$$

Clearly any trajectory originating in D, $\tilde{D}$, or $\tilde{\tilde{D}}$ remains within D for all $t \geq 0$, approaching the origin as $t \to \infty$. A trajectory originating in $\tilde{D}(\tilde{\tilde{D}})$, however, does not necessarily remain within $\tilde{D}(\tilde{\tilde{D}})$ although it cannot leave D.

Illustrative Example

A simple demonstration of the preceding method of constructing regions of attraction is provided by the system

$$\ddot{x} + a\dot{x} + bx + \phi(x) = 0 \tag{22}$$

$$(a > 0,\ b > 0)$$

Defining $x_1 = x$, $x_2 = \dot{x}$, as state variables,

$$A = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \underline{c} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{23}$$

$$G(i\omega) = \frac{1}{(b-\omega^2) + i\omega a}$$

and (3) is seen to be satisfied for infinite K provided $q = \frac{1}{a}$. Equation (8) then yields

$$\theta(i\omega) = \sqrt{b} \tag{24}$$

and, since $\gamma \triangleq -q\underline{c}'\underline{b} + 1/K = 0$, (9) implies

$$\underline{u} = \begin{bmatrix} \sqrt{b} \\ 0 \end{bmatrix} \tag{25}$$

The symmetric matrix B is found by (7) to be

$$B = \frac{1}{2a} \begin{bmatrix} a^2+b & a \\ a & 1 \end{bmatrix} \tag{26}$$

and therefore

$$\underline{c}'B^{-1}\underline{c} = \frac{2a}{b} \tag{27}$$

Consider first the class of all continuous odd non-linearities $\phi(\sigma)$ satisfying

$$0 < \frac{\phi(\sigma)}{\sigma} \quad \forall \sigma \varepsilon (\ell_1, \ell_2) \;, \qquad \sigma \neq 0 \tag{28}$$

$$\phi(0) = 0$$

for given ℓ_1, ℓ_2 , where $\ell_1 < 0 < \ell_2$. A region of attraction of the origin, valid for all such $\phi(\sigma)$, is given by the open connected region $\tilde{\tilde{D}}(0\varepsilon\tilde{\tilde{D}})$ all of whose interior points $\underline{x}$ satisfy

$$(ax_1 + x_2)^2 + bx_1^2 < b \min_{i=1,2} |\ell_i|^2 \tag{29}$$

It may be noted that the preceding choice of K, q, does not yield a nontrivial region $\tilde{D}$ which would be valid for non-odd nonlinearities. However a finite choice for K, and a correspondingly different matrix B, would provide such a region.

Consider now a specific nonlinearity, say

$$\phi(\sigma) = \sigma - \sigma^3 \tag{30}$$

for which $-\ell_1 = \ell_2 = 1$. Then a region of attraction $D(D \supset \tilde{\tilde{D}})$ is determined by

$$(ax_1 + x_2)^2 + (b + 1 - x_1^2/2)x_1^2 < b + 1/2 \tag{31}$$

Conclusions

The primary difficulty involved in determining these regions of attraction is seen to be in finding the polynomial θ. This requires finding the roots of an even polynomial with real coefficients of order $\leq 2n$, if (1) is of n^{th} order. This, however, is a problem which is subject to machine computation. Since the matrix B may then be determined by (7), relations (14)-(21) allow direct solution of the following problems:

1.) Given $\underline{c}$, $\underline{b}$, A, K, and $\phi(\sigma)$ for $\sigma \varepsilon (\ell_1, \ell_2)$, determine D.

2.) Given $\underline{c}$, $\underline{b}$, A, K, ℓ_1, and ℓ_2, determine $\tilde{D}$ and $\tilde{\tilde{D}}$.

3.) Given $\underline{c}$, $\underline{b}$, A, K, and $\tilde{M}_1$ or $\tilde{M}_2$, determine the permissable values of ℓ_1 and ℓ_2.

These are all significant practical problems since one seldom knows, in reality, how a nonlinearity actually behaves for large values of its argument. Although fairly tedious computations are involved in determining the regions described above, the method is straightforward and has the basic advantage of providing non-trivial regions of attraction for a wide class of problems.

Appendix A

Assume that assumptions (i) through (iv) hold and $\underline{u}$ is defined as in (8) and (9). Then (8) becomes

$$\gamma + \underline{m}^*(i\omega)\underline{h} + h'\underline{m}(i\omega) = (\sqrt{\gamma} - \underline{u}'\underline{m}(i\omega))(\sqrt{\gamma} - m^*(i\omega)\underline{u}) \quad (32)$$

where $2\underline{h} \triangleq - qA'\underline{c} - \underline{c}$, $\gamma \triangleq - q\underline{c}'\underline{b} + 1/K$, and $\underline{m}(z) \triangleq [zI - A]^{-1}\underline{b}$ with $\underline{m}^*(z)$ as its conjugate transpose. After some algebraic manipulation and the use of (7), (32) becomes

$$\text{Re } m^*(iw)(B\underline{b} - \underline{h} - \sqrt{\gamma}\,\underline{u}) = 0 \qquad \omega \geq 0 \quad (33)$$

and therefore

$$B\underline{b} - \sqrt{\gamma}\,\underline{u} = \underline{h} . \quad (34)$$

Consequently (6) is correct by direct computation.

To show that $B > 0$ note that (7) implies $B \geq 0$ since A is strictly Hurwitz, and consider a point $\underline{x}_1$ such that $B\underline{x}_1 = 0$. This implies, by (7), that $Be^{At}\underline{x}_1 \equiv 0$ and $\underline{u}'e^{At}\underline{x}_1 \equiv 0$. Therefore by (34) $\underline{h}'e^{At}\underline{x}_1 \equiv 0$, which implies that $\underline{x}_1 = 0$ if $(\underline{h}', A)$ is completely observable. To show that $(\underline{h}', A)$ is completely observable, note that by the definition of $\underline{h}$

$$2\underline{h}'[A-zI]^{-1}\underline{b} = - q\underline{c}'\underline{b} - (1+qz)\underline{c}'[A-zI]^{-1}\underline{b} \quad (35)$$

and therefore assumptions (ii), (iii), and (iv) imply that $\underline{h}'[A-zI]^{-1}\underline{b}$ is irreducible, which in turn implies that $(\underline{h}',A)$ is completely observable. Therefore B is positive definite.

Appendix B

The function $V(\underline{x})$ defined in (5) is clearly positive if $\sigma\varepsilon(\ell_1,\ell_2)$, $x \neq 0$, and $q \geq 0$, since $B > 0$ and $\phi(\sigma)$ is restricted by (2) in this region. However for $q < 0$ further investigation is required to verify (10).

Assume $q < 0$ and note that by (3) the plot of $G(i\omega)$, $\omega\varepsilon[0,\infty)$, does not intersect the real axis at, or to the left of, $-1/K$. Therefore, by the Nyquist criterion the equilibrium of system (1) is globally asymptotically stable for any linear characteristic $\phi(\sigma) = h\sigma$ such that $h\varepsilon[0,K]$. Also, by (6), $\dot{V}$ is non-positive for any such linear characteristic. Consequently $V(\underline{x})$ must be non-negative and, in particular,

$$\underline{x}'B\underline{x} + a\,\frac{K}{2}\,\sigma^2 \geq 0 \tag{36}$$

which implies

$$\underline{x}'B\underline{x} + q\int_0^\sigma \phi(\xi)d\xi > 0 \qquad \forall\sigma\varepsilon(\ell_1,\ell_2)\ ,\quad |\underline{x}| \neq 0 \tag{37}$$

and (10) is correct as stated.

References

M. A. Aizerman and F. R. Gantmacher, Absolute Stability of Regulator Systems, Holden-Day, Inc., 1964.

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